Pismeni dio ispita iz Matematike I

(strojari i građevinari)

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Zadatak 1. Odrediti kompleksan broj z iz uvjeta:

$$\left|\frac{z}{z+1}\right|=1, \quad \frac{z}{\bar{z}}=i,$$ pa zatim naći sva rješenja jednadžbe $A^3 = 2\cdot z$.

Zadatak 2. Riješiti sustav linearnih jednadžbi:

$$\begin{aligned} 2\cdot x_1 - 3\cdot x_2 + 4\cdot x_3 &= 7 \\ x_1 + 5\cdot x_2 - 3\cdot x_3 &= -1 \\ x_1 - x_2 + x_3 &= 5 \\ 3\cdot x_1 + 5\cdot x_2 - 5\cdot x_3 &= 15 \end{aligned}$$

Zadatak 3. Zadani su vektori:

$$\vec{a}=2\cdot\lambda\cdot\vec{i}+\vec{j}+(1-\lambda)\cdot\vec{k},\quad \vec{b}=-\vec{i}+3\cdot\vec{j},\quad \vec{c}=5\cdot\vec{i}-\vec{j}+8\cdot\vec{k}.$$

a) odrediti λ tako da vektor $\vec{a}$ gradi jednake kutove sa vektorima $\vec{b}$ i $\vec{c}$;

b) za nađeno λ naći površinu paralelograma konstruiranog nad vektorima $4\times\vec{a}$ i $\vec{b}$;

c) naći jednadžbu pravca koji prolazi točkom $M_1(1, 2, -3)$ paralelno vektoru $\vec{b}\times\vec{c}$.

Zadatak 4. Ispitati funkciju $y=\frac{x^3+1}{x^2}$, i nacrtati njen graf.

Zadatak 5. Naći realne brojeve a i b takve da funkcija $f(x)=\frac{\sin^2 x}{a-b\cdot\cos x}$ ima ekstremnu vrijednost $\frac{1}{4}$ za $x=\frac{\pi}{3}$.

Rješenja:

Zadatak 1. $\left|\frac{z}{z+1}\right|=1 \Leftrightarrow \frac{|z|}{|z+1|}=1 \Leftrightarrow \frac{\sqrt{x^2+y^2}}{\sqrt{(x+1)^2+y^2}}=1 \Leftrightarrow \sqrt{x^2+y^2}=\sqrt{(x+1)^2+y^2} \quad |^2$

$$x^2+\cancel{y^2}=(x+1)^2+\cancel{y^2} \quad \Leftrightarrow \quad \cancel{x^2}=\cancel{x^2}+2x+1 \quad \Leftrightarrow \quad 2x=-1 \quad \Leftrightarrow \quad \boxed{x=-\frac{1}{2}};$$

$$\frac{z}{\overline{z}}=i \quad \Leftrightarrow \quad z=\overline{z}\cdot i \quad \Leftrightarrow \quad x+y\cdot i=x\cdot i+y \quad \Leftrightarrow \quad y\cdot(i-1)=x\cdot(i-1) \quad \Leftrightarrow \quad \underline{\underline{y=x}}$$

$$\Rightarrow \quad \boxed{y=-\frac{1}{2}};$$

$$\Rightarrow \quad \boxed{z=-\frac{1}{2}-\frac{1}{2}\cdot i};$$

Sada vrijedi: $A^3=2\cdot z=2\cdot\left(-\frac{1}{2}-\frac{1}{2}\cdot i\right)=-1-i \quad \Rightarrow \quad A=\sqrt[3]{-1-i}$, pa ako ispod–korijensku veličinu označim kao $-1-i=z_1 \quad \Rightarrow \quad \rho_{z_1}=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$, i $\varphi_{z_1}=arctg\frac{-1}{-1}=\frac{5\cdot\pi}{4}$, jer je z_1 kompleksan broj iz trećeg kvadranta.

Sada je: $A=\sqrt[3]{\sqrt{2}}\cdot\left(\cos\frac{\frac{5\cdot\pi}{4}+2\cdot k\cdot\pi}{3}+i\cdot\sin\frac{\frac{5\cdot\pi}{4}+2\cdot k\cdot\pi}{3}\right), \quad k=0,\ 1,\ 2$

$$\Rightarrow A_0=\sqrt[6]{2}\cdot\left(\cos\frac{5\cdot\pi}{12}+i\cdot\sin\frac{5\cdot\pi}{12}\right),$$
$$A_1=\sqrt[6]{2}\cdot\left(\cos\frac{13\cdot\pi}{12}+i\cdot\sin\frac{13\cdot\pi}{12}\right),$$
$$A_2=\sqrt[6]{2}\cdot\left(\cos\frac{21\cdot\pi}{12}+i\cdot\sin\frac{21\cdot\pi}{12}\right).$$

Zadatak 2.

$$A_p = \left[\begin{array}{ccc:c} 2 & -3 & 4 & 7 \\ 1 & 5 & -3 & -1 \\ 1 & -1 & 1 & 5 \\ 3 & 5 & -5 & 15 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & -3 & 4 & 7 \\ 0 & 6 & -4 & -6 \\ 1 & -1 & 1 & 5 \\ 0 & 8 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & 3 & 0 & 1 \\ 0 & 6 & -4 & -6 \\ 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc:c} 2 & 3 & 0 & 1 \\ 0 & 3 & -2 & -3 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 3 & 0 & -9 \\ 0 & 3 & -2 & -3 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 3 & 0 & -9 \\ 0 & 0 & -2 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc:c} 0 & 1 & 0 & -3 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 1 & 0 & -3 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad r(A_p) = r(A) = 3 = n \quad \Leftrightarrow \quad \underline{\underline{\text{sustav ima jedinstveno rješenje.}}}$$

Iz posljednje matrice može se očitati jedinstveno rješenje sustava kao:

$x_1 = 5$,

$x_2 = -3$,

$x_3 = -3$.

Zadatak 3.

a) $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b})\Rightarrow\cos\measuredangle(\vec{a},\vec{b})=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

$\vec{a}\cdot\vec{c}=|\vec{a}|\cdot|\vec{c}|\cdot\cos\measuredangle(\vec{a},\vec{c})\Rightarrow\cos\measuredangle(\vec{a},\vec{c})=\dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$

$$\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c})\Leftrightarrow\frac{\vec{a}\cdot\vec{b}}{\cancel{|\vec{a}|}\cdot|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{\cancel{|\vec{a}|}\cdot|\vec{c}|}\Leftrightarrow\frac{-2\cdot\lambda+3}{\sqrt{1+9+0}}=\frac{10\cdot\lambda-1+8-8\cdot\lambda}{\sqrt{25+1+64}}\Leftrightarrow\frac{-2\cdot\lambda+3}{\sqrt{10}}=\frac{2\cdot\lambda+7}{\sqrt{90}}$$

$$\Leftrightarrow\frac{-2\cdot\lambda+3}{\sqrt{10}}=\frac{2\cdot\lambda+7}{3\sqrt{10}}\quad\Leftrightarrow\quad 3\cdot(-2\cdot\lambda+3)=2\cdot\lambda+7\quad\Leftrightarrow\quad -6\cdot\lambda+9=2\cdot\lambda+7\quad\Rightarrow\quad\boxed{\lambda=\frac{1}{4}};$$

b) $\vec{a}=\dfrac{1}{2}\cdot\vec{i}+\vec{j}+\dfrac{3}{4}\cdot\vec{k}\quad\Rightarrow\quad 4\cdot\vec{a}=2\cdot\vec{i}+4\cdot\vec{j}+3\cdot\vec{k},\quad\vec{b}=-\vec{i}+3\cdot\vec{j};$

$$P=|4\cdot\vec{a}\times\vec{b}|=\left|\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&4&3\\-1&3&0\end{vmatrix}\right|=|-9\cdot\vec{i}-3\cdot\vec{j}+10\cdot\vec{k}|=\sqrt{81+9+100}=\boxed{\sqrt{190}};$$

c) $M_1(1,\ \ 2,\ \ -3)\in p,\quad p\,||\,\vec{b}\times\vec{c};$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-1&3&0\\5&-1&8\end{vmatrix}=24\cdot\vec{i}+8\cdot\vec{j}-14\cdot\vec{k}\quad\Rightarrow\quad\boxed{p\ldots\frac{x-1}{12}=\frac{y-2}{4}=\frac{z+3}{-7}}.$$

Zadatak 4.

1° Domena: $D(f) = \mathbb{R} \setminus \{0\}$;

2° Nule funkcije: $x^3 + 1 = 0$ => $x^3 = -1$ => $x = -1$ je nula funkcije;

3° V.A.

$$\lim_{x \to 0^+} \frac{x^3+1}{x^2} = +\infty, \quad \lim_{x \to 0^-} \frac{x^3+1}{x^2} = +\infty \quad \Rightarrow \quad x = 0 \text{ je V.A.}$$

4° K.A. / H.A.

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3+1}{x^3} = 1,$$

$$l = \lim_{x \to \infty} \left(f(x) - k \cdot x \right) = \lim_{x \to \infty} \left(\frac{x^3+1}{x^2} - x \right) = \lim_{x \to \infty} \left(\frac{x^3+1-x^3}{x^2} \right) = \frac{1}{x^2} = 0 \quad \Rightarrow \quad y = x \text{ je K.A.}$$

Nema H.A. jer je $\lim_{x \to \infty} f(x) = \infty$;

5° Stacionarne točke: $y' = \frac{3 \cdot x^4 - 2 \cdot x^4 - 2 \cdot x}{x^4} = \frac{x^4 - 2 \cdot x}{x^4} = \frac{x \cdot (x^3 - 2)}{x^4} = \frac{x^3 - 2}{x^3}$,

$$y' = 0 \quad \Leftrightarrow \quad x^3 - 2 = 0 \quad \Leftrightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2} \text{ je stacionarna točka;}$$

6° $D(y') = \mathbb{R} \setminus \{0\}$;

7° Točke infleksije i ekstremne točke:

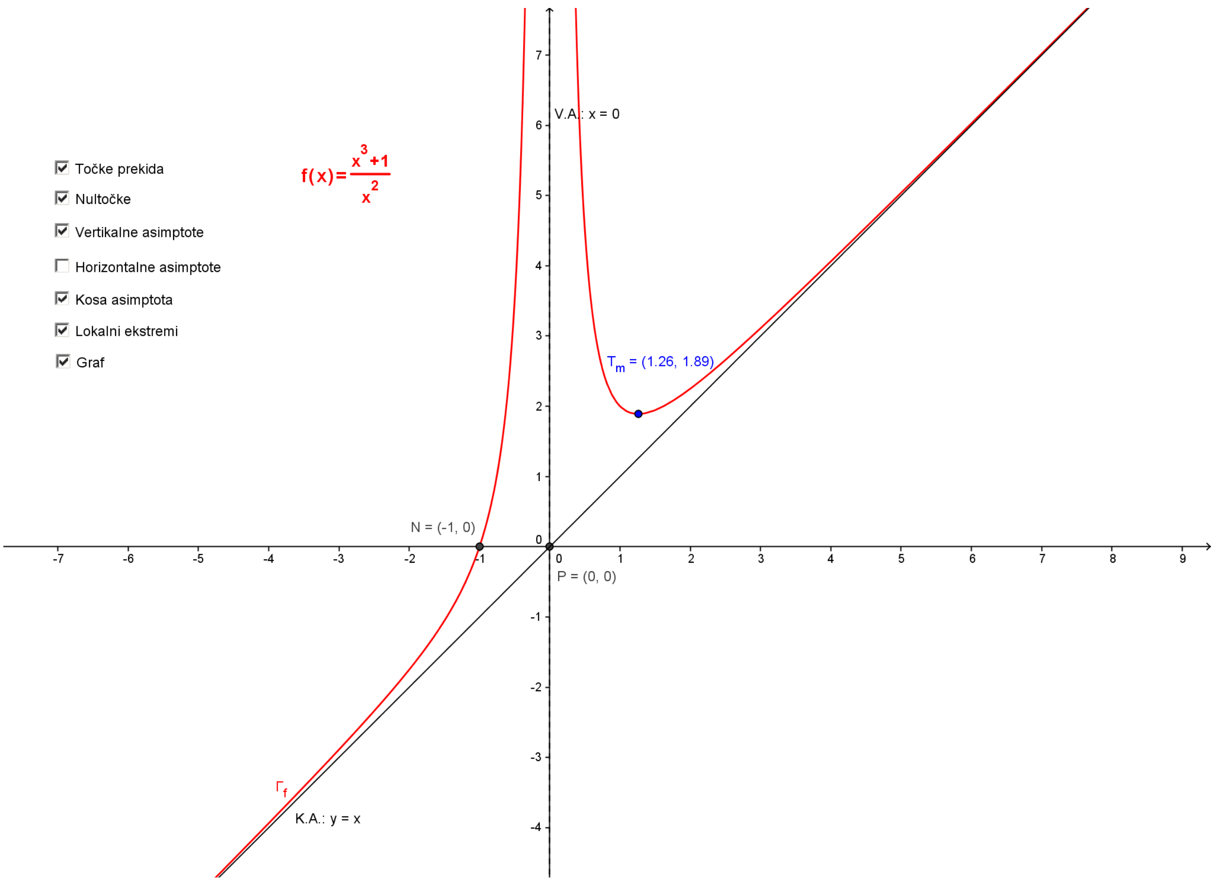
$$y'' = \frac{3 \cdot x^5 - 3 \cdot x^5 + 6 \cdot x^2}{x^6} = \frac{6 \cdot x^2}{x^6} = \frac{6}{x^4} > 0 \quad \forall x \quad \Rightarrow$$ funkcija nema točaka infleksije;

Nadalje, vrijedi: $y''\left(\sqrt[3]{2}\right) > 0 \quad \Rightarrow \quad T\left(\sqrt[3]{2}, f\left(\sqrt[3]{2}\right)\right) \equiv T\left(\sqrt[3]{2}, \frac{3}{\sqrt[3]{4}}\right)$ je točka lokalnog minimuma;

8° Tok funkcije:

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,\sqrt[3]{2})$	$\sqrt[3]{2}$	$(\sqrt[3]{2},+\infty)$
y''	+	+	+		+	+	+
y'	+	+	+		–	**0**	+
y	↗	**0**	↗		↘	**min.**	↗
	∪	∪	∪		∪	∪	∪

9° Graf funkcije:



Zadatak 5. $a,\ b=?$ tako da $\dfrac{1}{4}$ bude ekstremna vrijednost funkcije

$$f(x)=\frac{\sin^2 x}{a-b\cdot\cos x} \quad \text{za} \quad x=\frac{\pi}{3};$$

Vrijedi da je: $f\left(\frac{\pi}{3}\right)=\frac{1}{4}$, pa je

$$f\left(\frac{\pi}{3}\right)=\frac{\sin^2\frac{\pi}{3}}{a-b\cdot\cos\frac{\pi}{3}}=\frac{1}{4} \Leftrightarrow \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{a-b\cdot\frac{1}{2}}=\frac{1}{4} \Leftrightarrow \frac{\frac{3}{4}}{a-\frac{b}{2}}=\frac{1}{4} \Leftrightarrow 3=a-\frac{b}{2} \Leftrightarrow \boxed{b=2\cdot(a-3)};$$

$$f'(x)=\frac{2\cdot\sin x\cdot\cos x\cdot(a-b\cdot\cos x)+\sin^2 x\cdot b\cdot\sin x}{(a-b\cdot\cos x)^2}=\frac{2\cdot a\cdot\sin x\cdot\cos x-2\cdot b\cdot\sin x\cdot\cos^2 x+b\cdot\sin^3 x}{(a-b\cdot\cos x)^2}$$

$$f'\left(\frac{\pi}{3}\right)=0 \Leftrightarrow \frac{2\cdot a\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}-2\cdot b\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{4}+b\cdot\frac{3\sqrt{3}}{8}}{\left(a-b\cdot\frac{1}{2}\right)^2}=0 \Leftrightarrow \frac{a\cdot\sqrt{3}}{2}-\frac{b\cdot\sqrt{3}}{4}+\frac{3\cdot b\cdot\sqrt{3}}{8}=0 \Leftrightarrow$$

$$\Leftrightarrow 4\cdot a-2\cdot b+3b=0 \Leftrightarrow \boxed{b=-4a} \Leftrightarrow \boxed{a=1,\quad b=-4}.$$