# Derivacije

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

**Tablica osnovnih derivacija:**

1. $(c)' = 0$,
2. $(x)' = 1$,
3. $(x^n)' = nx^{n-1}$,
4. $(a^x)' = a^x \cdot \ln a$,
5. $(e^x)' = e^x$,
6. $(\ln x)' = \frac{1}{x}$, **(x > 0)**
7. $(\log_a x)' = \frac{\log_a e}{x}$, **(x > 0, a > 0, a ≠ 1)**
8. $(\sin x)' = \cos x$,
9. $(\cos x)' = -\sin x$,
10. $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$,
11. $(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$,
12. $(\text{arc sin } x)' = \frac{1}{\sqrt{1-x^2}}$, **(|x| < 1)**
13. $(\text{arc cos } x)' = -\frac{1}{\sqrt{1-x^2}}$, **(|x| < 1)**
14. $(\text{arc tg } x)' = \frac{1}{1+x^2}$,
15. $(\text{arc tg } x)' = -\frac{1}{1+x^2}$.

**Osnovna pravila deriviranja:**

1. $(c \cdot f)' = c \cdot f'$
2. $(f \pm g) = f' \pm g'$
3. $(f \cdot g)' = f' \cdot g + f \cdot g'$
4. $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

**Zadaci:**

Derivacije funkcija zadanih eksplicitno:

1. $y = x^4 - 3x^3 + 5x^2 + 6x - 7$ $[y' = 4x^3 - 9x^2 + 10x + 6]$
2. $y = (x - a) \cdot (x - b)$ $[y' = 2x - a - b]$
3. $y = e^x \cdot (x^2 - 4x + 5)$ $[y' = (x - 1)2 \cdot e^x]$
4. $y = \frac{1 + \ln x}{x}$ $\left[y' = -\frac{1}{x^2} \ln x\right]$
5. $y = \frac{1}{2}(x + \sin x \cos x)$ $[y' = \cos^2 x]$

6. $y = \sqrt{1+\sqrt[3]{x}}$ $\left[ y' = \frac{1}{6\sqrt[3]{x^2}\sqrt{1+\sqrt[3]{x}}} \right]$

7. $y = (\cos 3x)^5$ $[y' = -15\cos^4 3x \cdot \sin 3x]$

8. $y = (x \cdot \sin x)^3$ $[y' = 3(x \cdot \sin x)^2(\sin x + x \cdot \cos x)]$

9. $y = \ln \operatorname{ctg} \frac{x}{2}$ $\left[ y' = -\frac{1}{\sin x} \right]$

10. $y = \sqrt[3]{\sin 2x}$ $\left[ y' = \frac{2\cos 2x}{3\sqrt[3]{\sin^2 2x}} \right]$

11. $y = \frac{e^x}{2}(\sin x - \cos x)$ $[y' = e^x \cdot \sin x]$

12. $y = 2x \cdot \sin x + (2 - x^2) \cdot \cos x$ $[y' = x^2 \cdot \sin x]$

13. $y = \frac{1}{2}(x^2+1) \cdot \operatorname{arctg} x - \frac{1}{2}(x+1)$ $[y' = x \cdot \operatorname{arctg} x]$

14. $y = x \cdot (\sin \ln x - \cos \ln x)$ $[y' = 2\sin \ln x]$

15. $y = \frac{x^3}{3}\left( \ln x - \frac{1}{3} \right)$ $[y' = x^2 \cdot \ln x]$

16. $y = x\sqrt{1+x^2}$ $\left[ y' = \frac{1+2x^2}{\sqrt{1+x^2}} \right]$

17. $y = \frac{1}{3}(x^2+2)\sqrt{x^2-1}$ $\left[ y' = \frac{x^3}{\sqrt{x^2-1}} \right]$

18. $y = x \cdot \operatorname{tg} x + \ln \cos x$ $\left[ y' = \frac{x}{\cos^2 x} \right]$

19. $y = -\frac{1}{2}(x^2+1) \cdot e^{-x^2}$ $[ y' = x^3 e^{-x^2} ]$

20. $y = \ln\left( e^x + \sqrt{1+e^{2x}} \right)$ $\left[ y' = \frac{e^x}{\sqrt{1+e^{2x}}} \right]$

21. $y = 2\left( \sin\sqrt{x} - \sqrt{x}\cos\sqrt{x} \right)$ $\left[ y' = \sin\sqrt{x} \right]$

22. $y = x(\arcsin x)^2 + 2\sqrt{1-x^2}\cdot \arcsin x - 2x$ [y' = (arc sin x)²]

23. $y = \ln\left(\sin x + \sqrt{1+\sin^2 x}\right)$ $\left[y' = \frac{\cos x}{\sqrt{1+\sin^2 x}}\right]$

24. $y = \frac{1}{2}\left(x + \sqrt{1-x^2}\right)e^{\arcsin x}$ [y' = e^{arc sin x}]

25. $y = \ln\sqrt{\frac{1-\sin x}{1+\sin x}}$ $\left[y' = -\frac{1}{\cos x}\right]$

26. $y = \ln\sqrt{\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}}$ $\left[y' = \frac{1}{\cos 2x}\right]$

27. $y = x\cdot\ln^2 x - 2x\cdot(\ln x - 1)$ [y' = ln² x]

28. $y = x\cdot \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)$ [y' = arc tg x]

29. $y = \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)$ $\left[y' = \frac{1}{\cos x}\right]$

30. $y = \frac{3x}{8} - \left(\frac{1}{4}\sin^3 x + \frac{3}{8}\sin x\right)\cdot\cos x$ [y' = sin⁴ x]

31. $y = \ln\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$ $\left[y' = -\frac{2}{\cos 2x}\right]$

32. $y = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x$ $\left[y' = \sqrt{1-x^2}\right]$

33. $y = x\cdot\arcsin x + \sqrt{1-x^2}$ [y' = arc sin x]

34. $y = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}$ $\left[y' = \frac{1}{\sqrt{1-x^2}}\right]$

35. $y = \operatorname{arctg}\left(x - \sqrt{1+x^2}\right)$ $\left[y' = \frac{1}{2\sqrt{1+x^2}}\right]$

36. $y = \operatorname{arctg}\frac{x+1}{x-1}$ $\left[y' = -\frac{1}{x^2+1}\right]$

37. $y = \ln\sqrt{\operatorname{tg}\frac{x}{2}}$ $\left[y' = \frac{1}{2\sin x}\right]$

38. $y = \ln\sqrt{\frac{1+x}{1-x}}$ $\left[y' = \frac{1}{1-x^2}\right]$

39. $y = \text{arc sin}(3x - 4x^3)$ $\left[y' = \frac{3}{\sqrt{1-x^2}}\right]$

40. $y = \sqrt[x]{x}$ $\left[y' = \sqrt[x]{x^{1-2x}}(1-\ln x)\right]$

41. $y = (1+x)^x$ $\left[y' = (1+x)^x\left[\ln(1+x) + \frac{x}{1+x}\right]\right]$

42. $y = (\ln x)^{\frac{1}{x}}$ $\left[y' = \frac{1}{x^2}(\ln x)^{\frac{1}{x}}\left[\frac{1}{\ln x} - \ln(\ln x)\right]\right]$

43. $y = x^x$ $[y' = x^x(\ln x + 1)]$

44 $y = x^{\sin x}$ $\left[y' = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)\right]$

## Derivacije funkcija zadanih implicitno:

Funkciju zadanu u obliku $F(x, y) = 0$ deriviramo na slijedeći način:

a) deriviramo obe strane jednadžbe $F(x, y) = 0$ po varijabli $x$ uzimajući da je $y = y(x)$,

b) dobivenu jednakost $\frac{d}{dx}F(x, y)$ rješimo po $y'$.

45. $x^3 + y^3 + 2xy - 1 = 0$ $\left[y' = -\frac{3x^2 + 2y}{3y^3 + 2x}\right]$

46. $x + y + e^{xy} - 1 = 0$ u $M(0, 1)$ $\left[y' = -\frac{1 + ye^{xy}}{1 + xe^{xy}};\quad y'(0,1) = 2\right]$

47. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ $\left[y' = \frac{b^2 x}{a^2 y}\right]$

48. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $\left[y' = -\sqrt[3]{\frac{y}{x}}\right]$

49. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $\left[ y' = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y} \right]$

50. $y = 1 + e^{xy}$ $\left[ y' = \frac{ye^{xy}}{1 - xe^{xy}} \right]$

51. $x = y \cdot e^{\sin y}$ $\left[ y' = \frac{e^{-\sin y}}{1 + y\cos y} \right]$

52. $\operatorname{arctg}\frac{y}{x} = \ln\sqrt{x^2 + y^2}$ $\left[ y' = \frac{x + y}{x - y} \right]$

53. $y^3 - 2xy^2 + 1 = 0$ $\left[ y' = \frac{2y}{3y - 4x} \right]$

54. $y = e^{x^2 + y^2}$ $\left[ y' = \frac{2xy}{1 - 2y^2} \right]$

## Derivacije funkcija zadanih parametarski:

Ako je funkcija zadana parametarski jednadžbama $x = \varphi(t)$ i $y = \psi(t)$ tada je:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{\varphi'(t)}$$

55. $\left.\begin{aligned} x &= r\cos t \\ y &= r\sin t \end{aligned}\right\}$ $[y' = -\operatorname{ctg} t]$

56. $\left.\begin{aligned} x &= a(t - \sin t) \\ y &= a(1 - \cos t) \end{aligned}\right\}$ $\left[ y' = \operatorname{ctg}\frac{t}{2} \right]$

57. $\left.\begin{aligned} x &= a\cos^3 t \\ y &= b\sin^3 t \end{aligned}\right\}$ $\left[ y' = -\frac{b}{a}\operatorname{tg} t \right]$

58. $\left.\begin{aligned} x &= a(2\cos t - \cos 2t) \\ y &= a(2\sin t - \sin 2t) \end{aligned}\right\}$ $\left[ y' = \operatorname{tg}\frac{3t}{2} \right]$

59. $\left.\begin{aligned} x &= \cos t \cdot \sin^2 t \\ y &= \sin t(1 + \cos^2 t) \end{aligned}\right\}$ $[y' = \operatorname{ctg} t]$

60. $\left.\begin{aligned} x &= \frac{3at}{1+t^3} \\ y &= \frac{3at^2}{1+t^3} \end{aligned}\right\}$ $\left[ y' = \frac{t(2-t^3)}{1-2t^3} \right]$

61. $\left.\begin{aligned} x &= e^t \sin t \\ y &= e^t \cos t \end{aligned}\right\}$ $\left[ y' = \frac{\cos t - \sin t}{\cos t + \sin t} \right]$

62. $\left.\begin{aligned} x &= \sin t\sqrt{\cos 2t} \\ y &= \cos t\sqrt{\cos 2t} \end{aligned}\right\}$ [y' = – tg 3t]

Primjena derivacija:

Tangenta krivulje $y = f(x)$ u točki $M(x_m, y_m)$ ima jednadžbu $y - y_m = -y'(x_m)\cdot(x - x_m)$.

Normala krivulje $y = f(x)$ u točki $M(x_m, y_m)$ ima jednadžbu $y - y_m = \frac{1}{y'(x_m)}\cdot(x - x_m)$.

[$y'(x_m)$ je vrijednost derivacije funkcije $y = f(x)$ za $x = x_m$]

63. U točki $x = 1$ nađi jednadžbu tangente i normale krivulje:
   a) $y = x^3 - 3x^2 + 2x + 1$
      [$y = -x + 2$; $y = x$]
   b) $y = \frac{1}{x^2}$
      [$y = -2x + 3$; $x - 2y + 1 = 0$]
   c) $y = \ln x$
      [$y = -x + 1$]

64. Nađi jednadžbu tangente krivulje $y = x^2 + 2$ koja je paralelna s pravcem $y = x - 2$.
   [$4x - 4y + 7 = 0$]

65. Na krivulji $y = x^2 - 2x + 1$ nađi točku u kojoj je normala paralelna s pravcem
   $x + 2y - 3 = 0$.
   [$x = 2$]

66. Iz ishodišta su povučene tangente na krivulju $y = ax^2 + bx + c$. Nađi koordinate dodirnih točaka.
   $\left[ x = \pm\sqrt{\frac{c}{a}} \right]$

67. Naći jednadžbu tangente na krivulju:
   a) $y^3 - 2x^2y + a^3 = 0$ u točki M(a, a)
   [y = 4x – 3a]
   b) $x - y = \ln(1 + 2x + y)$ u točki O(0, 0)
   $\left[ y = -\frac{1}{2}x \right]$
   c) $y + e^{xy} - 2 = 0$ u točki P(0, 1)
   [y = –x + 1]

68. Naći jednadžbu normale krivulje:
   a) $y = x + \ln(2x - y)$ u točki M(1, 1)
   [2x + 3y – 5 = 0]
   b) $y^2 = \sin(x + y)$ u točki O(0, 0)
   [y = x]
   c) $y = 1 + x \cdot e^y$ u točki N(–1, 0)
   [y = –2(x + 1)]

69. Naći jednadžbu tangente na krivulju:
   a) $\left.\begin{matrix} x = a\cos t \\ y = b\sin t \end{matrix}\right\}$ u točki $t = \frac{\pi}{4}$
   $\left[ \frac{x}{a} + \frac{y}{b} = \sqrt{2} \right]$
   b) $\left.\begin{matrix} x = e^{-t} \\ y = t\cos t \end{matrix}\right\}$ u točki t = 0
   [y = –x + 1]
   c) $\left.\begin{matrix} x = t^2 \\ y = t^3 \end{matrix}\right\}$ u točki t = 1
   [3x – 2y – 1 = 0]

70. Naći jednadžbu normale krivulje:
   a) $\left.\begin{matrix} x = e^{-t} \\ y = \ln(t+1) \end{matrix}\right\}$ u točki t = 0
   [y = x – 1]
   b) $\left.\begin{matrix} x = t \cdot e^t \\ y = t \cdot \ln t \end{matrix}\right\}$ u točki t = 1
   [y = –2e(x –e)]

71. Naći kut pod kojim se sijeku krivulje $x^2 + y^2 = 8$ i $y = \frac{1}{2}x^2$.
   [φ = arc tg 3]

72. U točki P(1, 2) krivulje $y = x^3 + 1$ povučene je tangenta i ona siječe krivulju u točki Q. Naći koordinate točke Q.
[Q(–2, –7)]

73. Iz točke A(–1, 0) povučena je tangenta ne krivulju $y^2 = x^3 + 1$ i ona dodiruje krivulju u točki B. Naći koordinate točke B.
$\left[ B_1(-1+\sqrt{3}, \sqrt{6\sqrt{3}-9});\ B_2(-1+\sqrt{3}, -\sqrt{6\sqrt{3}-9}) \right]$

74. U točki M(–1, 0) krivulje $y = x^4 + 7x^3 + 17x^2 + 16x + 5$ povučena je tangenta koja siječe danu krivulju u točkama A i B. Izračunati duljinu tetive $\overline{AB}$.
$\left[ \overline{AB} = \sqrt{2} \right]$

75. Tangenta krivulje $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ siječe koordinatne osi u točkama A i B. Pokazati da je $\overline{AB}$ = const.
$\left[ \overline{AB} = a \right]$

76. Pokazati da krivulja $y = \dfrac{x^4 - 5x^2 + 4}{4x^3 - 10x}$ siječe os Ox uvijek pod istim kutom i naći taj kut.
$\left[ \alpha = \dfrac{\pi}{4} \right]$

77. Normale povučene u točkama $x = 0$ i $x = a$ krivulje $y = e^x$ sijeku se u točku C. Naći granični položaj točke C kad $a \to 0$.
[C(–2, 3)]

78. Pokazati da je svaka normala krivulje $\left.\begin{aligned} x &= a(\cos t + t \cdot \sin t) \\ y &= a(\sin t - t \cdot \cos t) \end{aligned}\right\}$ tangenta kružnice $\left.\begin{aligned} x &= a \cdot \cos t \\ y &= a \cdot \sin t \end{aligned}\right\}$.

79. Tangenta krivulje $y = \dfrac{a^2}{2x}$ siječe koordinatne osi i točkama A i B. Pokazati da je površina trokuta OAB konstantna.