Pismeni dio ispita iz Matematike I

izvanredni ispitni rok

(strojari i građevinari)

Mostar, 14. listopad 2003.

Zadatak 1. Naći produkt svih rješenja jednadžbe $z^4 - \sin\frac{2\pi}{3} + i\cos\frac{\pi}{3} = 0$.

Rješenja unijeti u Gaussovu ravninu.

Zadatak 2. Riješiti matričnu jednadžbu $(X \cdot A - E) \cdot B = (B - E) \cdot B$ ako je

$A = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \\ 1 & -3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 11 & 1 & 0 \\ -12 & 2 & 3 \end{bmatrix}$, a E je jedinična matrica.

Zadatak 3. Dana je ravnina $3x - y + 2z - 6 = 0$. Naći:

a) površinu trokuta kojeg iz dane ravnine isjecaju koordinatne osi;

b) prodor pravca koji prolazi ishodištem i stoji okomito na danu ravninu.

Zadatak 4. Ispitati funkciju $y = \frac{12 - 6x}{(x+1)^2}$ i nacrtati njen graf.

Zadatak 5. Odrediti realne brojeve a i b tako da funkcija $f(x) = \frac{1}{x^2 + ax + b}$

ima maksimum u točki $T\left(\frac{3}{2}, -4\right)$.

Rješenja:

Zadatak 1.

$$z^4 - \sin\frac{2\pi}{3} + i\cos\frac{\pi}{3} = 0 \quad \Leftrightarrow \quad z^4 = \sin\frac{2\pi}{3} - i\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2}i \quad \Leftrightarrow \quad z = \sqrt[4]{\frac{\sqrt{3}}{2} - \frac{1}{2}i}\,.$$

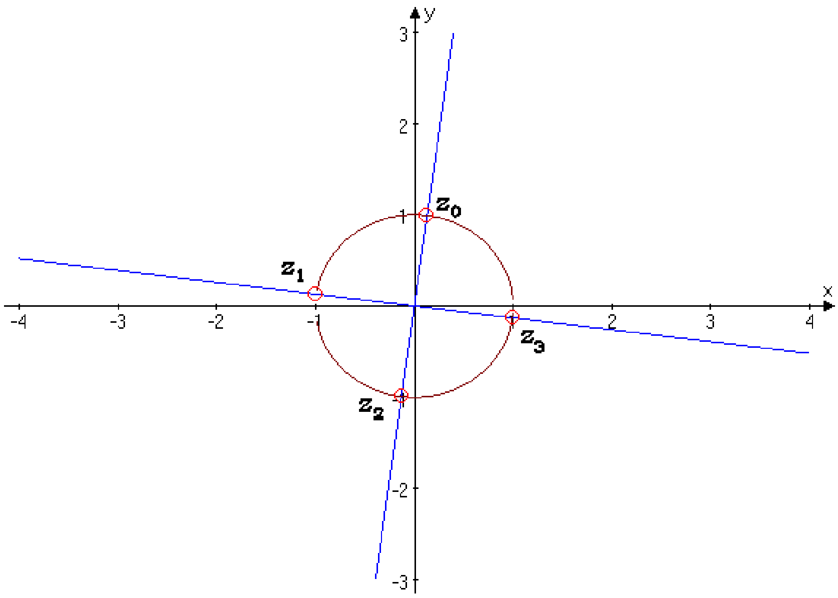
$$\left|\frac{\sqrt{3}}{2} - \frac{1}{2}i\right| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1\,,$$

$$arctg\,\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = arctg\,\frac{-1}{\sqrt{3}} = arctg\,\frac{-\sqrt{3}}{3} = \frac{11\pi}{6}\,.$$

$$\Rightarrow \quad z = \sqrt[4]{1}\left(cos\,\frac{\frac{11\pi}{6} + 2k\pi}{4} + i\sin\frac{\frac{11\pi}{6} + 2k\pi}{4}\right), \quad k = 0,\ 1,\ 2,\ 3 \quad \Rightarrow$$

$$\Rightarrow \begin{cases} z_0 = \cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24} \\ z_1 = \cos\frac{23\pi}{24} + i\sin\frac{23\pi}{24} \\ z_2 = \cos\frac{35\pi}{24} + i\sin\frac{35\pi}{24} \\ z_3 = \cos\frac{47\pi}{24} + i\sin\frac{47\pi}{24} \end{cases} \quad \Rightarrow \quad z_0 \cdot z_1 \cdot z_2 \cdot z_3 = \cos\frac{116\pi}{24} + i\sin\frac{116\pi}{24} = \cos\frac{29\pi}{6} + i\sin\frac{29\pi}{6}$$

$$\Rightarrow \quad z_0 \cdot z_1 \cdot z_2 \cdot z_3 = \cos\,(4\pi + \frac{5\pi}{6}) + i\sin\,(4\pi + \frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i\,.$$



Zadatak 2.

$(X \cdot A - E) \cdot B = (B - E) \cdot B \,\big|\cdot B^{-1}$

$X \cdot A - \not{E} = B - \not{E}$

$X \cdot A = B \,\big|\cdot A^{-1}$

$\boxed{X = B \cdot A^{-1}}$

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \\ 1 & -3 & -5 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \\ 1 & -3 & -5 \end{vmatrix}\begin{matrix} 2 & -1 \\ 3 & 2 \\ 1 & -3 \end{matrix} = -20 - 1 + 18 + 4 + 6 - 15 = -8,$$

$A_{11} = -7, \quad A_{12} = 16, \quad A_{13} = -11, \quad A_{21} = 1, \quad A_{22} = -8, \quad A_{23} = 5, \quad A_{31} = 3, \quad A_{32} = -8, \quad A_{33} = 7$

$$\Rightarrow \quad A^{-1} = -\frac{1}{8} \cdot \begin{bmatrix} -7 & 1 & 3 \\ 16 & -8 & -8 \\ -11 & 5 & 7 \end{bmatrix} \Rightarrow \quad X = -\frac{1}{8} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 11 & 1 & 0 \\ -12 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -7 & 1 & 3 \\ 16 & -8 & -8 \\ -11 & 5 & 7 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad X = -\frac{1}{8} \cdot \begin{bmatrix} -18 & 6 & 10 \\ -61 & 3 & 25 \\ 83 & -13 & -31 \end{bmatrix}.$$

Zadatak 3.

$3x - y + 2z - 6 = 0$

a) $3x - y + 2z = 6 \quad \big| : 6$

$\boxed{\frac{x}{2} + \frac{y}{-6} + \frac{z}{3} = 1} \Rightarrow A(2,\ 0,\ 0), \quad B(0,\ -6,\ 0), \quad C(0,\ 0,\ 3),$

a to su točke presjeka dane ravnine s koordinatnim osima.

Sada je: $\overrightarrow{AB} = (-2,\ -6,\ 0)$, $\overrightarrow{AC} = (-2,\ 0,\ 3)$, pa vrijedi:

$$P_{\Delta} = \frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{2} = \frac{1}{2} \cdot \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -6 & 0 \\ -2 & 0 & 3 \end{matrix} \right\| = \frac{1}{2} \cdot \left| -18\vec{i} + 6\vec{j} - 12\vec{k} \right| = \frac{1}{2} \cdot \sqrt{(-18)^2 + 6^2 + (-12)^2} =$$

$$= \frac{\sqrt{504}}{2} = \boxed{3 \cdot \sqrt{14}}.$$

b) Evidentno je kako vektor normale na danu ravninu glasi: $\vec{n}=(3,\ -1,\ 2)$,

pa je jednadžba pravca koji je okomit na danu ravninu a prolazi

ishodištem zadana sa: $p\ldots\ \frac{x}{3}=\frac{y}{-1}=\frac{z}{2}$;

Sada možemo uzeti:

$$\frac{x}{3}=\frac{y}{-1} \ \Leftrightarrow\ -x=3y \ \Leftrightarrow\ \boxed{x=-3y},\quad \text{i}\quad \frac{y}{-1}=\frac{z}{2}\ \Leftrightarrow\ \boxed{z=-2y},$$ što

uvrštavajući u jednadžbu dane ravnine dobivamo:

$$3(-3y)-y+2(-2y)-6=0 \ \Rightarrow\ \boxed{y=-\frac{3}{7}} \ \Rightarrow\ \boxed{x=\frac{9}{7}} \ \Rightarrow\ \boxed{z=\frac{6}{7}};$$

Sada je tražena točka prodora: $\boxed{P\left(\frac{9}{7},\ -\frac{3}{7},\ \frac{6}{7}\right)}$.

Zadatak 4. $y=\frac{12-6x}{(x+1)^2}$

1° $D(f)=\mathbb{R}\setminus\{-1\}$;

2° Nule funkcije:

$f(x)=0$ **<=>** 12 – 6x = 0 => $\boxed{x = 2 \text{ je nula funkcije;}}$

3° (Ne)parnost funkcije: $f(-x)=\frac{12-6(-x)}{(-x+1)^2}\neq \pm f(x)$ => funkcija ni parna ni neparna;

4° V.A. $\lim\limits_{x\to -1^{+0}}\frac{12-6x}{(x+1)^2}=+\infty$, $\lim\limits_{x\to -1^{-0}}\frac{12-6x}{(x+1)^2}=+\infty \ \Rightarrow\ \boxed{x=-1}$ je V.A.

K.A. / H.A.

$\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\frac{12-6x}{(x+1)^2}=0 \ \Rightarrow\ \boxed{y=0}$ je H.A. $\Rightarrow$ nema K.A.

5° Stacionarne točke:

$$y' = \frac{-6(x+1)^2 - 2(x+1)(12-6x)}{(x+1)^4} = \frac{-2(x+1)\left[3(x+1)+12-6x\right]}{(x+1)^4} = \frac{-2(3x+3+12-6x)}{(x+1)^3} =$$

$$= \frac{-2(15-3x)}{(x+1)^3};$$

$y' = 0 \quad \Leftrightarrow \quad 15-3x=0 \quad \Leftrightarrow \quad x=5 \quad \Rightarrow \quad \boxed{x=5}$ je stacionarna točka.

6° $D(f') = \mathbb{R} \setminus \{-1\};$

7° Točke infleksije i ekstremne točke:

$$y'' = \frac{6(x+1)^3 + 2(15-3x)\cdot 3(x+1)^2}{(x+1)^6} = \frac{6(x+1)^2\left[x+1+15-3x\right]}{(x+1)^6} = \frac{6(16-2x)}{(x+1)^4};$$

$y'' = 0 \quad \Leftrightarrow \quad 16-2x=0 \quad \Leftrightarrow \quad x=8 \quad \Rightarrow \quad \boxed{x=8}$ je točka infleksije;

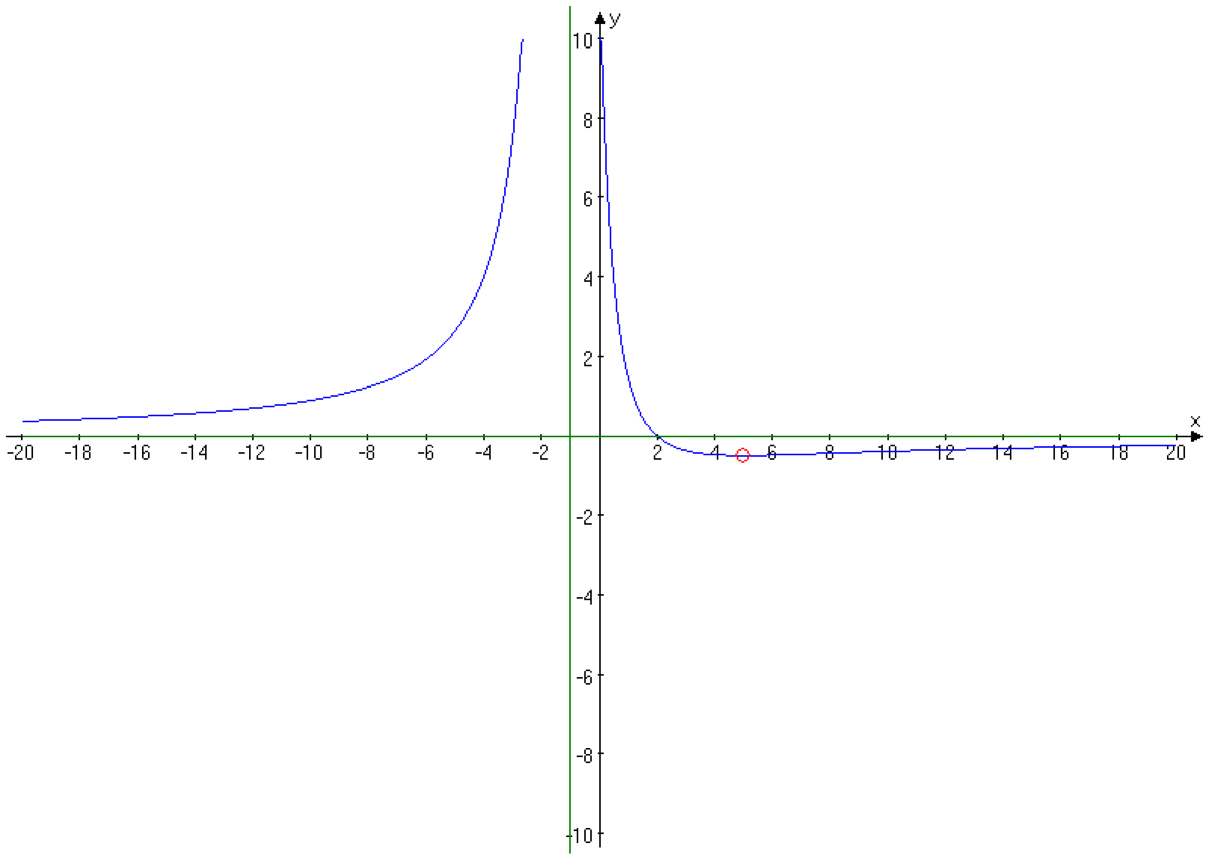
$y''(5) = \dfrac{6(16-10)}{6^4} > 0 \quad \Rightarrow \quad x=5$ je točka lokalnog minimuma, i vrijedi

$\boxed{y_{\min} = -\dfrac{1}{2}}$.

8° Tok funkcije:

x	$-\infty$	-1	2	5	8	$+\infty$
y''	+	+	+	+	**0**	−
y'	+	−	−	**0**	+	+
y	↗, ∪	↘, ∪ **0**	↘, ∪	**min** ↗, ∪	**inf** ↗, ∩	

9° Graf funkcije:



<u>Zadatak 5.</u> a, b = ?, $f(x)=\dfrac{1}{x^2+ax+b}$ ima maksimum u točki $T\left(\dfrac{3}{2},\ -4\right)$;

$$f'(x)=\frac{0-(2x+a)}{(x^2+ax+b)^2}=\frac{-2x-a}{(x^2+ax+b)^2} \Rightarrow f'(x)=0 \Leftrightarrow \boxed{-2x-a=0};$$

Kako je T točka lokalnog maksimuma, vrijedi: $f'\left(\dfrac{3}{2}\right)=0 \Rightarrow -2\dfrac{3}{2}-a=0 \Rightarrow$

$\Rightarrow \boxed{a=-3};$

Evidentno je da vrijedi:

$$f\left(\frac{3}{2}\right)=\frac{1}{\frac{9}{4}+\frac{3}{2}a+b}=-4 \Rightarrow \frac{1}{\frac{9+6a+4b}{4}}=-4 \Rightarrow \boxed{9+6a+4b=-1} \Rightarrow 4b=-10-6a$$

$\Rightarrow 4b=-10+18=8 \Rightarrow \boxed{b=2}$.